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Lesson 1: Integer Sequences—Should You Believe in Patterns?

Student Outcomes

* Students examine sequences and are introduced to the notation used to describe them.

Lesson Notes

A sequence in high school is simply a function whose domain is the positive integers. This definition will be given later in the module (after functions have been defined). In this first lesson, we are setting the stage for an in–depth study of sequences by allowing students to become acquainted with the notation. For now, we will use the description: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

For example, (P, O, O, L) is a sequence that is different than (L, O, O, P). Usually the terms are *indexed* (and therefore ordered) by a subscript starting at 1: . The “…” symbol indicates that the pattern described is regular; that is, the next term is , the next is , and so on. In the first example, = ‘P’ is the first term, = ‘O’ is the second term, and so on. Infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of = … can be thought of as being represented by the sequence , , , , …. Sequences (and series) are an important part of studying calculus.

In general, a sequence is defined by a function from a domain of positive integers to a range of numbers that can be either integers or real numbers (depending on the context) or other non-mathematical objects that satisfies the equation . When that function is expressed as an algebraic function only in terms of numbers and the index variable , then the function is called the *explicit form of the sequence (or explicit formula)*. For example, the function , which satisfies for all positive integers , is the explicit form for the sequence .

**Important:** Sequences can be indexed by starting with any integer. For example, the sequence can be indexed by , … by stating the explicit formula as for . This can create real confusion for students about what the “fifth term in the sequence” is: In the list, the 5th term is , but by the formula, the 5th term could mean . To avoid such confusion, in this module ***we adopt the convention that indeces start at 1***. That way the first term in the list is always or , and there is no confusion about what the 100th term is. Students are, however, exposed to the idea that the index can start at a number other than . The lessons in this topic also offer suggestions about when to use and when to use .

Classwork

Opening Exercise (5 minutes)

After reading through the task, ask students to discuss part (a) with a partner; then share responses as a class. Next, have students answer parts (b) and (c) in pairs before discussing as a class.

* Because the task provides no structure, all of these answers must be considered correct. Without any structure, continuing the pattern is simply speculation—a guessing game.
* Because there are infinitely many ways to continue a sequence, the sequence needs to provide enough structure to define, say, the 5th, 10th, and 100th terms.
* A sequence can be thought of as an ordered list of elements. If you believe a sequence of numbers is following some structure or pattern, then it would be nice to have a formula for it.

*Scaffolding:*

Challenge early finishers to come up with other possible patterns for this sequence.

Opening Exercise

Mrs. Rosenblatt gave her students what she thought was a very simple task:

What is the next number in the sequence 2, 4, 6, 8, …?

*Cody: I am thinking of a “plus 2 pattern,” so it continues 10, 12, 14, 16, ….*

*Ali: I am thinking a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, ….*

*Suri: I am thinking of the units digits in the multiples of two, so it continues 2, 4, 6, 8, 0, 2, 4, 6, 8, ….*

1. Are each of these valid responses?

Each response must be considered valid because each one follows a pattern.

1. What is the hundredth number in the sequence in Cody’s scenario? Ali’s? Suri’s?

Cody: 200 Ali: 8 Suri: 0

1. What is an expression in terms of for the th number in the sequence in Cody’s scenario?

is one example. Note: Another student response might be if the student starts with (see Example 1).

Example 1 (5 minutes)

The focus of this example should be on the discussion of whether to start the sequence with or , a concept that can be very challenging for students. Allow students a few minutes to consider the example independently before discussing the example as a class.

**The main point of this example**: Even though there is nothing wrong with starting sequences at (or any other integer for that matter), ***we will agree that during this module we will always start our sequences at .*** That way, the th number in the list is the same as the *n*th term in the sequence, which corresponds to or or or whatever formula name we give the *n*th term.

* Some of you have written and some have written Which is correct? Allow students to debate which is correct before saying…
* Is there any way that both could be correct?
  + If we started by filling in 0 for , then is correct. If we started with , then is correct.
* Get into a discussion about starting the sequence with an index of one versus an index of zero. Either formula is correct. Thus, a decision must be made when writing a formula for a sequence as to whether to start the term number at 0 or 1. Generally, it feels natural to start with 1, but are there cases where it would feel more natural to start with 0?
  + *Yes! If the sequence is denoting values changing over time, it often makes sense to start at time 0 rather than a time of 1. Computer programmers start with term 0 rather than term 1 when creating Javascript arrays. The terms in a polynomial begin with 0.*

**MP.7**

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**MP.8**

Your goal as the teacher in having this discussion is to acknowledge and give validation to students who started with 0 and wrote . Explain to the class that there is nothing wrong with starting at 0, but to make it easier for us to communicate about sequences during this module, we will adopt the convention that during this module we will always start our sequences at .

* Use the discussion as an opportunity to connect the term number with the term itself as well as the notation by using the following visual:

Start with this table: Then, lead to this table:

|  |  |
| --- | --- |
| Sequence Term | Term |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |



|  |  |
| --- | --- |
| Term Number | Term  New Notation: |
|  |  |
|  |  |
|  |  |
|  |  |
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|  |  |

* Students are already familiar with the an notation. Let this lead to the f(n) notation as shown outside of the second table above. Emphasize how each of these is spoken. The third term of a sequence could be called “a sub 3” or “f of 3.”

Example 1

Jerry has thought of a pattern that shows powers of two. Here are the first 6 numbers of Jerry’s sequence:

…

*Scaffolding:*

For students who are struggling, assist them in writing out a few calculations to determine the pattern:

…

Write an expression for the th number of Jerry’s sequence.

The expression generates the sequence starting with .

Example 2 (8 minutes)

We are introducing function notation right away but without naming it as such and without calling attention to it at this stage. The use of the letter for *formula* seems natural, and the use of parentheses does not cause anxiety or difficulty at this level of discussion. Watch to make sure students are using the to stand for *formula for the th term* and not thinking about it as the product .

* Should we always clarify which value of we are assuming the formula starts with?
  + *Unless specified otherwise, we are assuming for this module that all formulas generate the sequence by starting with .*
* Are all the points on a number line in the sequence?
  + *No, the graph of the sequence consists of only the discrete dots (not all the points in between).*

Example 2

Consider the sequence that follows a “plus 3” pattern: ….

* 1. Write a formula for the sequence using both the notation and the notation.

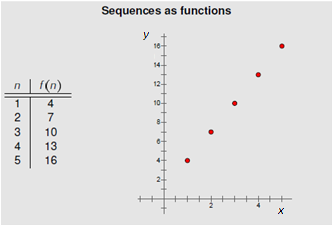
starting with .

* 1. Does the formula generate the same sequence? Why might some people prefer this formula?

Yes. . It is nice that the first term of the sequence is a term in the formula, so one can almost read the formula in plain English: Since there is the “plus 3’” pattern, the nth term is just the first term plus that many more threes.

* 1. Graph the terms of the sequence as ordered pairs on the coordinate plane. What do you notice about the graph?

The points all lie on the same line.

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Exercises 1–5 (17 minutes)

Allow students time to work on the exercises either individually or in pairs. Circulate the room assisting students when needed, especially with writing the formula. Then, debrief by sharing answers as a class.

After students work individually, ask the following:

* If we had instead used the formula to generate the sequence … by starting at , how would we find the 3rd term in the sequence? The 5th term? The *n*th term?

Point out to students that by choosing to start all formulas at , we do not need to worry about how to compute the nth term of a sequence using the formula. For example, by using the formula , we can compute the 150th term simply by finding the value .

Make sure the students are making discrete graphs rather than continuous.

* Were any of the graphs linear? How do you know?
  + *Example 2 was linear because it has a constant rate of change (subtract 5 each time).*

Exercises 1–5

1. Refer back to the sequence from the Opening Exercise. When Dr. T was asked for the next number in the sequence …, he said . ?

Yes, using the formula,

* 1. Does his formula actually produce the numbers and ?

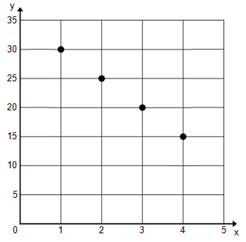
Yes.

* 1. What is the 100th term in the Dr. T’s sequence?

1. Consider a sequence that follows a “minus 5” pattern: ….
   1. Write a formula for the nth term of the sequence. Be sure to specify what value of your formula starts with.

if starting with .

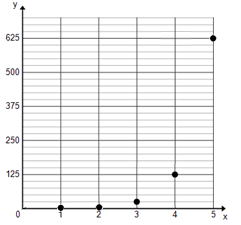
* 1. Using the formula, find the 20th term of the sequence.
  2. Graph the terms of the sequence as ordered pairs on a coordinate plane.



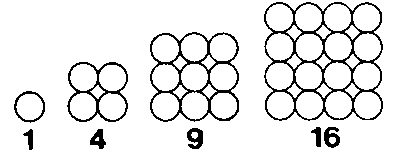
1. Consider a sequence that follows a “times 5” pattern: ….
   1. Write a formula for the th term of the sequence. Be sure to specify what value of your formula starts with.

if starting with .

* 1. Using the formula, find the 10th term of the sequence.
  2. Graph the terms of the sequence as ordered pairs on a coordinate plane.



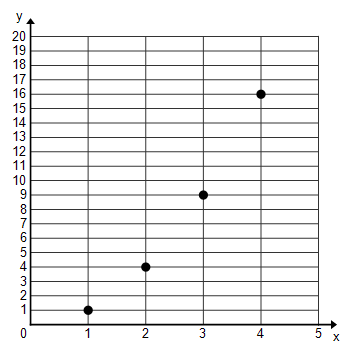
1. Consider the sequence formed by the square numbers:



* 1. Write a formula for the nth term of the sequence. Be sure to specify what value of n your formula starts with.

starting with .

* 1. Using the formula, find the 50th term of the sequence.
  2. Graph the terms of the sequence as ordered pairs on a coordinate plane.

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1. A standard letter-sized piece of paper has a length and width of 8.5 inches by 11 inches.
   1. Find the area of one piece of paper.

square inches

* 1. If the paper were folded completely in half, what would be the area of the resulting rectangle?

square inches

**MP.4**

* 1. Write a formula for a sequence to determine the area of the paper after folds.

starting with , OR starting with .

* 1. What would the area be after 7 folds?

square inches

Closing (5 minutes)

* Why is it important to have a formula to represent a sequence?
  + *It is important to have a formula to represent a sequence to demonstrate the specific pattern and to help in finding any term of the sequence (20th term, 50th term, 1000th term, etc.). This is very useful when we need to make predictions based on the formula chosen to model the existing data sequence.*
* Can one sequence have two different formulas?
  + *Yes, depending on what value of n you chose to start with. Some students may point out that two different-looking formulas may be related by equivalent expressions: is different than .*
* What does represent? How is it read alound?
  + *It represents the nth term of a sequence just like . It is read “of .”*

Lesson Summary

A sequence can be thought of as an ordered list of elements. To define the pattern of the sequence, an explicit formula is often given, and unless specified otherwise, the first term is found by substituting 1 into the formula.

Exit Ticket (5 minutes)

Name \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Date\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Lesson 1: Integer Sequences—Should You Believe in Patterns?

Exit Ticket

1. Consider the sequence given by a “plus 8” pattern: …. Shae says that the formula for the sequence is . Marcus tells Shae that she is wrong because the formula for the sequence is .   
   1. Which formula generates the sequence by starting at ? At ?

* 1. Find the 100th term in the sequence.

1. Write a formula for the sequence of cube numbers: ….

Exit Ticket Sample Solutions

1. Consider the sequence given by a “plus 8” pattern: ….

Shae says that the formula for the sequence is . Marcus tells Shae that she is wrong because the formula for the sequence is .

* 1. Which formula generates the sequence by starting at ? At ?

Shae’s formula generates the sequence by starting with , while Marcus’s formula generates the sequence by starting with .

* 1. Find for each formula.

Marcus’s formula.

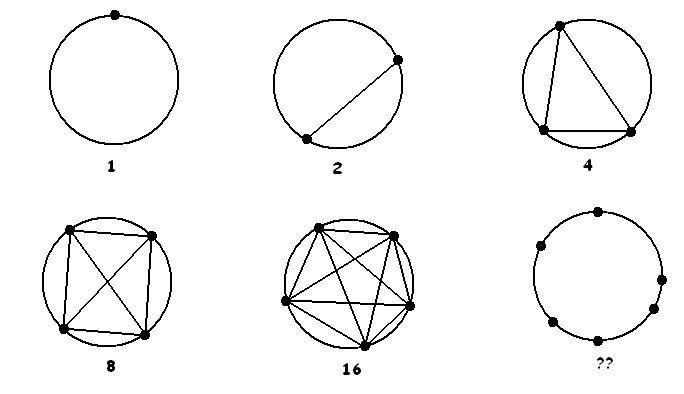
Shae’s formula: .

1. Write a formula for the sequence of cube numbers: ….

starting with .

Problem Set Sample Solutions

1. Consider a sequence generated by the formula starting with . Generate the terms and .
2. Consider a sequence given by the formula starting with . Generate the first 5 terms of the sequence.
3. Consider a sequence given by the formula starting with . Generate the first 5 terms of the sequence.
4. Here is the classic puzzle that shows that patterns need not hold true. What are the numbers counting?

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The number under each figure is counting the number of (non-overlapping) regions in the circle formed by all the segments connecting all the points on the circle. Each graph contains one more point on the circle than the previous graph.

* 1. Based on the sequence of numbers, predict the next number.
  2. Write a formula based on the perceived pattern.

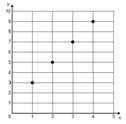
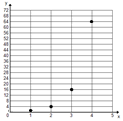
starting with .

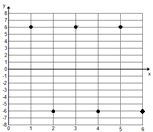
* 1. Find the next number in the sequence by actually counting.
  2. Based on your answer from (c), is your model from b effective for this puzzle?

No, it works for to but not for . And we don’t know what happens for values of larger than 6.

In problems 5-8, for each of the following sequences:

* 1. Write a formula for the th term of the sequence. Be sure to specify what value of your formula starts with.
  2. Using the formula, find the 15th term of the sequence.
  3. Graph the terms of the sequence as ordered pairs (,) on a coordinate plane.

1. The sequence follows a “plus 2” pattern: ….
   1. starting with .
   2. 
2. The sequence follows a “times 4” pattern: ….
   1. stating with .
   2. 

1. The sequence follows a “times -1” pattern: ….
   1. starting with .
   2. 

1. The sequence follows a “minus 3” pattern: ….
   1. starting with .
   2. 